Math Circles - Intro to Combinatorics - Winter 2024

Lecture 1

February 7th, 2024

1 Background

We begin by reviewing factorials.

Definition 1.0.1 *For $n \in \mathbb{N}$ n-factorial, denoted $n!$,* $n! = \prod_{i=1}^{n} i = n \times (n-1) \times ... \times 1$.

Note we define $0! = 1$. Why? This is a standard definition for the empty product. We will point out another reason this is nice later.

Problems 1.0.2 *Try the following:*

1. $5!$
2. $5 \times 4!$
3. $2!3!$
4. $\frac{5!}{3!}$
5. $\frac{4!}{(3-1)!}$
6. $\frac{6!}{4!2!}$

Do you notice anything interesting about your answers to one and two? What about four, five, and six?

Solution 1.0.3 *Notice that one and two have the same solution and the last three problems are are whole numbers.*

1. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
2. $5 \times 4! = 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$
3. $2!3! = (2 \times 1)(3 \times 2 \times 1) = 2 \times 6 = 12$
4. $\frac{5!}{3!} = \frac{120}{6} = 20$ *or* $\frac{5!}{3!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 5 \times 4 = 20$
5. $\frac{4!}{(3-1)!} = \frac{4!}{2!} = \frac{4 \times 3 \times 2}{2} = 12$
6. $\frac{6!}{4!2!} = \frac{6 \times 5}{2} = 15$

2 Introduction to Permutations and Combinations

So what is combinatorics? In the simplest sense it is the theory of counting. But how do we count more complex things? In order to do this, we use different rules. We will start by looking at some problems to help us discover what some of these rules are.

2.1 Basic Counting Rules

Example 2.1.1 *Suppose you are packing for a trip and you are only going to wear shorts and t-shirts while you are gone. If you pack 4 shirts and 3 pairs of pants, how many different outfits can you wear while you are traveling, supposing we don't care about shoes, socks, or accessories?*

Let's try and count this. We will build all possible outfits.

shirt 1 and pants 1	*shirt 1 and pants 2*	*shirt 1 and pants 3*
shirt 2 and pants 1	*shirt 2 and pants 2*	*shirt 2 and pants 3*
shirt 3 and pants 1	*shirt 3 and pants 2*	*shirt 3 and pants 3*
shirt 4 and pants 1	*shirt 4 and pants 2*	*shirt 4 and pants 3*
shirt 5 and pants 1	*shirt 5 and pants 2*	*shirt 5 and pants 3*

We can build 15 outfits. What do you notice about this number? Well, $15 = 5 \times 3$ *and we had* 5 *options for the shirt and* 3 *for the pants. We will generalize this pattern.*

Theorem 2.1.2 *(The Fundamental Counting Principal) If there are* m *ways to make a choice and then* n *ways to make a second choice after, there are* $m \times n$ *total ways to make both choices.*

Remark 2.1.3 *This idea extends to what we call the* Product Rule *of counting. Which is if some number of independent choices are made we can find the total number of choices by multiplying the number of options for each choice together. Notice the order of the choices doesn't matter.*

What if the order of our choices does matter? Consider the following example.

Example 2.1.4 *Suppose you trying to pick a gift for a friends birthday. You want to get your friend a specific customized t-shirt. There are two different patterns to choose from. The first pattern comes with a background in pink, purple, red, or blue and the second pattern comes with the background in green, blue, or black. How many different options do you have?*

Let's build a table again to figure out.

Pattern 1 in pink	*Pattern 2 in green*
Pattern 1 in purple	*Pattern 2 in blue*
Pattern 1 in red	*Pattern 2 in black*
Pattern 1 in blue	

We can build 7 different shirts. Which is 4+3 or the number of choices for the first option plus the number of options for the second. This generalizes as well.

Remark 2.1.5 *This idea extends to what we call the* Sum Rule *of counting. Which is if some number of sequential choices are made, where the order of choices matters or are not independent, we can find the total number of choices by adding the number of options for each choice together.*

2.2 Practice Problems

Problems 2.2.1 *Try the following:*

1. *Suppose you have 6 markers, 8 colored pencils, and 4 highlighters. How many ways can you pick one of each?*

2. *What if instead of picking one of each in the previous problem you need to pick one writing implement. How many ways can you pick this single writing implement?*

3. *How many different three letter words(where a word is just a string of letters) can you make with a,b, and c? No repeats allowed.*

4. *Do the previous question where you are allowed to repeat letters.*

5. *How many two digit numbers can you make that only contain the digits 3,7, and 8?*

Solution 2.2.2 *Each problem needs either the sum or product rule.*

1. *We have* $6 \times 8 \times 4 = 192$ *choices using the product rule.*

2. *We have* $6 + 8 + 4 = 18$ *choices using the sum rule.*

3. *Since we don't allow repeats we have* $3 \times 2 \times 1 = 6$ *choices.*

4. *Since repeats are allowed we have three choices for each letter. So* $3 \times 3 \times 3 = 3^3 = 27$ *words.*

5. *Two digit numbers allow repeated digits so we have three choices for each digit. Hence* $3 \times 3 = 9$.

2.3 Permutations

We are now going to talk about one of the two main types of object counting in combinatorics.

Definition 2.3.1 *A* permutation *is a selection of objects in a fixed order.*

Example 2.3.2 *Suppose we want to arrange a set of cups in a row. Each way that the cups can be arranged is a permutation.*

Some important properties of permutations are that we do not allow repetition(unless explicitly stated) and the order of the objects is what matters.

So how do we count permutations? Suppose we have n objects we want to arrange. Well we have n choices for the first object, $n-1$ choices for the second object, and so on until we have 1 choice for the final object. Now the product rule tells us we can multiply theses choices together. So we have $n(n-1)(n-2)...(2)(1)$ total ways to arrange n objects. Does this look familiar? It's $n!$. The number of ways to arrange n objects is $n!$.

What if I don't want to arrange all of my objects? What if I have n objects and just want to arrange r where $r < n$ of them? Then I have n choices for the first object, $n-1$ for the second and so on until I have $n-r$ choices or the last object. So I have $n(n-1)...(n-r+1)$ choices total by the product rule. Can we write this as a factorial? Yes! We can write this as $\frac{n!}{n-r!}$ since we don't want the last r objects.

Let's look at this in formal mathematical language.

Definition 2.3.3 *A permutation of a set* r *object selected from* n *total objects, denoted* $P(n,r) = \frac{n!}{n-r!}$.

If we want a permutation of zero objects, we have exactly one way to achieve this. So it makes sense for 0! to be 1.

Remark 2.3.4 *What if we want to allow repetition? Then we have* n *choices every time, so we have* n^r *total options.*

Problems 2.3.5 *Consider the following examples.*

- *You have ten books and a shelf big enough for five books. How many different ways can you put five books on the self.*

- *How many different permutations of the letters in KING ARTHUR are there?*

- *How many ways can you elect a president, vice president, a secretary, and a treasurer from ten people? What if a person can hold multiple positions?*

- *How many ways can you arrange* 7 *garden gnomes where two gnomes are part of a set and have to be next to each other. What if we want the pair of gnomes split up?*

Solution 2.3.6 *We will use permutations for these questions.*

- $P(10,5) = 30,240$

- *We want to arrange all letters which is* $P(10,10) = 10!$*, but there are two r's which are not distinguishable. So we have to divide by the number of ways to order the two r's. Which is* $P(2,2) = 2!$*. Thus we have* $\frac{10!}{2!} = 1814400$ *total ways.*

- *We want to pick four people, supposing we pick them in the order of president, vice president, secretary, and treasurer this is* $P(10,4) = 5040$*. If we allow people to hold multiple positions we have ten choices to elect at each position and thus* 10^4 *options.*

- *There are* $6!$ *ways to arrange the* 7 *gnomes if we could the pair as a single object to place. Now for each arrangement we have two ways to order our paired gnomes and thus* $2 \times 6!$ *total ways to arrange the gnomes. If we want the two gnomes separated we can use the classic trick of taking the total number of gnome arrangements* 7 *and subtracting the number of arrangements we don't want, which we just found. So we have* $7! - 2 \times 6!$ *arrangements where the gnomes are not together.*

2.4 Combinations

What if the order we are choosing our objects in doesn't matter?

Definition 2.4.1 *A* Combination *is a selection of objects in which the order they are chosen doesn't matter.*

Example 2.4.2 *If you have ten different mugs and want to make five mugs of tea, the number of ways you can pick five mugs to use if a combination.*

So how can we could this? Well if we have n objects and want a combination of all of them we have one option: to pick them all. What happens then if we only want to pick $r < n$ objects? Well we could use our permutation formula to pick r objects in order, but then we have to get rid of the order. We need to figure out how to tell if two permutations contain the same set of objects? Well if we have a set of r objects we know that there are $P(r,r)$ ways to rearrange a set of r objects. So for every set of r objects we choose from the n objects, $P(r,r)$ of the ways are equivalent. So we have $\frac{P(n,r)}{P(r,r)} = \frac{n!0!}{(n-r)!r!} = \frac{n!}{r!(n-r)!}$.

In formal mathematical language we have,

Definition 2.4.3 *A combination of a set* r *object chosen from* n *total objects, denoted* $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Notice that $C(n,r) = C(n, n-r)$

Remark 2.4.4 *What if we want to allow repetition? This is a bit more complicated than for permutations. Consider the following illustrative example.*

Suppose you want to buy twelve doughnut and there are three different, flavors, chocolate strawberry, and vanilla. How many different selections can you make? So $n = 3$ *and* $r = 12$

You could pick all twelve to be the same, or only choose two flavors of even use all three. How can we account for this?

Since order doesn't matter, we will suppose that we pick the number of chocolate first, then strawberry, and then the rest will be vanilla. So we have twelve "slots" and we want to divide them into three groups. This is similar to placing two "dividers" into fourteen slots. So we have $C(14,2) = C(14,12)$ *ways to do this.*

Generalizing this idea, if we are selecting r *objects from* n *elements set with repetition allowed, there are* $C(n+r-1, r) = C(n+r-1, n-1)$ *ways to do so.*

Problems 2.4.5 *Consider the following examples.*

- *Suppose you are given a list of five problems and are asked to turn in solutions to three problems. How many different choices can you make?*

- *At a cafeteria you can pick four items from: pasta, fried rice, salad, tomato soup, burger, brownie, cookie, banana, and apple. How many days can you have lunch before you will have to order the exact same set of things as a previous day, supposing you can order more than one serving or any time.*

- *Suppose a school is putting together a five person team for a math competition. If there are 127 eligible seniors and 97 eligible juniors and the team must have more seniors than juniors, how many different teams can the school make?*

Solution 2.4.6 *We will use combinations for these questions.*

- $\binom{5}{3} = \frac{5!}{2!3!} = \frac{5\times 4}{2} = 10.$
- *There are nine total menu items and we are allowing repetition. So you have* $\binom{9+4-1}{4} = \frac{12!}{4!8!} = \frac{12(11)(10)9}{4(3)(2)} = 495$ *different menus. So you can go 495 times before you will have to repeat a combination.*
- *We have three different options if we have to have more seniors. We can have five, four, or three seniors. So we will use the sum rule on these options. Thus we have* $\binom{127}{5} + \binom{127}{4}\binom{97}{1} + \binom{127}{3}\binom{97}{2}$*. We will leave the answer in this form.*